Assignment 12.

This homework is due *Thursday*, December 6.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems is due December 7.

Problems 8 and 9 are optional, meaning that they will only go to the numerator of your grade (not because they are any harder than other problems, but rather because this assignment looks a bit too large). Altogether, you can go as high as 140% on this homework.

Please have in mind that this homework cannot be extended past December 7, except for medical (or other valid) reasons.

(1) (a) (9.3.32i) For a nonempty subset E of a metric space (X, ρ) and a point $x \in X$, define the distance from x to E, $\operatorname{dist}(x, E)$ as follows:
$$\operatorname{dist}(x, E) = \inf\{\rho(x, y) \mid y \in E\}.$$
Show that the distance function $f : X \to \mathbb{R}$ defined by $f(x) = \operatorname{dist}(x, E)$, for $x \in X$, is continuous.

(b) (9.3.32ii) Show that $\{x \in X \mid \operatorname{dist}(x, E) = 0\} = \overline{E}$.

(c) (9.3.34) Show that a subset E of a metric space X is closed if and only if there is a continuous function $f : X \to \mathbb{R}$ for which $E = f^{-1}(0)$.

(d) (9.3.33) Show that a subset E of a metric space X is open if and only if there is continuous function $f : X \to \mathbb{R}$ for which $E = \{x \in X \mid f(x) > 0\}$.

(2) (9.4.38) In a metric space X, show that a Cauchy sequence converges if and only if it has a convergent subsequence.

(3) ($\sim$9.4.39) Let $0 < \alpha < 1$. Suppose that $\{x_n\}$ is a sequence in a complete metric space (X, ρ) and for each n, $\rho(x_n, x_{n+1}) \leq \alpha^n$. Show that $\{x_n\}$ converges. Does $\{x_n\}$ necessarily converge if we only require that for each n, $\rho(x_n, x_{n+1}) \leq 1/n$?

(4) (10.3.33) Let (X, ρ) be a *compact* metric space and T a mapping $X \to X$ such that
$$\rho(T(u), T(v)) < \rho(u, v) \text{ for all } u, v \in X, u \neq v.$$
Show that T has a unique fixed point. (*Hint:* Show that if there are no fixed points, the function $\rho(T(u), T^2(u))/\rho(u, T(u))$ from X to $\mathbb{R}$ is continuous and therefore reaches its maximum. Then follow the proof of Banach Contraction Principle using Problem 3.)

The problems below can be found in the Section 10.2 of textbook.

(5) (a) Recall that in Problem 5 of Homework 11 we defined interior $\operatorname{int} E$, exterior $\operatorname{ext} E$ and boundary $\operatorname{bd} E$ of a subset E of a metric space. Show that for every subset E of a metric X, $X = \operatorname{int} E \cup \operatorname{ext} E \cup \operatorname{bd} E$ and the union is disjoint.

(b) Recall that a subset A of a metric space X is called *dense* in X if every nonempty open subset of X contains a point of A. Further, a subset of a metric space X is called hollow in X if it has empty interior. Show that for a subset E of a metric space X, E is hollow in X if and only if $X \setminus E$ is dense in X.

— see next page —

(6) Prove the following theorem:
(*The Baire Category Theorem.*) Let X be a complete metric space. Let $\{\mathcal{O}_n\}$ be a countable collection of open dense subsets of X. Then the intersection $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ also is dense.
(*Hint:* You need to show that an arbitrary open ball $B(x_0, r_0)$ contains a point of $\bigcap_{n=1}^{\infty} \mathcal{O}_n$. Start by saying that $B(x_0, r_0) \cap \mathcal{O}_1$ is nonempty (why) and open (why), therefore contains an open ball $B(x_1, r_1)$ and a smaller closed[1] ball $B_1 = \overline{B}(x_1, r_1/2)$. Repeat argument with the open ball $B(x_1, r_1/2)$ and $\mathcal{O}_2$, and so on. Get a descending sequence of closed balls $B_1, B_2, \ldots$. Apply the Cantor Intersection Theorem.)

(7) Prove the following theorem:
(*The Baire Category Theorem.*) Let X be a complete metric space. Let $\{F_n\}$ be a countable collection of closed hollow subsets of X. Then the union $\bigcup_{n=1}^{\infty} F_n$ is also hollow.
(*Hint:* Apply Problem 5b to the assertion of Problem 6.)

(8) Let X be a complete metric space and $\{F_n\}$ a countable collection of closed subsets of X. If $\bigcup_{n=1}^{\infty} F_n$ has nonempty interior (for example, if $\bigcup_{n=1}^{\infty} F_n = X$), prove that at least one of the F_n's has nonempty interior.
(*Hint:* Pass to appropriate closed subset of X. Use Problem 7.)
The above result is also called Baire Category Theorem.

(9) Prove the following theorem.
Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X that is pointwise bounded in the sense that for each $x \in \mathbb{X}$, there is a constant M_x for which

$$|f(x)| \leq M_x \text{ for all } f \in \mathcal{F}.$$

Then there is nonempty open subset $\mathcal{O}$ of $\mathbb{X}$ on which $\mathcal{F}$ is uniformly bounded in the sense that there is a constant M for which

$$|f| \leq M \text{ on } \mathcal{O} \text{ for all } f \in \mathcal{F}.$$

(*Hint:* Define $E_n = \{x \in X : \ |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$. Use Problem 8.)

1. Extra Problems

(10) (10.2.20) Let F_n be the subset of $C[0,1]$ consisting of functions for which there is a point x_0 in $[0,1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.
  (a) Show that F_n is closed.
  (b) Show that F_n is hollow. (*Hint:* Show that for $f \in C[0,1]$ and $r > 0$, there is a piecewise linear "saw-like" function $g \in C[0,1]$ for which $\rho_\infty(f, g) < r$ and the left-hand and right-hand derivatives of g on $[0,1]$ are greater than $n+1$.)
  (c) Conclude by Baire Category theorem that $C[0,1] \neq \bigcup_{n=1}^{\infty} F_n$.
  (d) Show that each $h \in C[0,1] \setminus \bigcup_{n=1}^{\infty} F_n$ is not differentiable at any point in $[0,1]$. (*Hint:* If f is differentiable at x_0 and continuous on $[0,1]$, then $|f(x) - f(x_0)|/|x - x_0|$ is bounded "close" to x_0 by differentiability, and bounded "far" from x_0 by boundedness of f on $[0,1]$; so it belongs to some F_n.)

Note. Congratulations, you proved that there are continuous functions on $[0,1]$ that are not differentiable *anywhere*. Moreover, you proved that the set of such functions is *dense* in $C[0,1]$.

(11) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and have derivatives of all orders. Suppose that for each $x \in \mathbb{R}$, there is index $n = n(x)$ for which $f^{(n)}(x) = 0$. Show that f is a polynomial. (*Hint:* Use Baire Category Theorem.)
Comment. If you know that n is the same for all x, the statement easily follows by calculus.

---

[1] A closed ball $\overline{B}(x, r)$ is the set $\{y \in X \mid \rho(x, y) \leq r\}$. Show that it is a closed set.